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A FIXED POINT THEOREM FOR λ-GENERALIZED CONTRACTION OF METRIC SPACES

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ABSTRACT
In this paper we prove a fixed point theorem for λ-generalized contractions and obtain its consequences.

KEYWORDS: *D*-metric space,K-contraction,* $\lambda-generalized\ contraction$.

I. INTRODUCTION
In this paper we prove a fixed pint theorem for λ-generalized contractions and obtain its consequences. Also we define (ε,λ)-uniformly locally generalized contractions and prove a fixed point theorem for such contractions in the present paper.Fixed point theorems were established for self maps of metric spaces. Certain fixed point theorems were proved for self maps of metrizable topological spaces also since such spaces, for all practical purposes, can be considered as metric spaces.

Recently in Dhage [1] has initiated a study of general metric spaces called *D*-metric spaces. Later several researchers have made a significant contribution to the fixed point theorems of D-metric In a different way R. Kannan [2] has defined a contraction for metric spaces which we shall call a K-contraction. Analogously we define the K-contractions for *D**-metric spaces. As a probable modification of D-metric spaces, very recently, Shaban Sedghi, Nabi Shobe and Haiyun Zhou [3] have introduced D*-metric spaces.

II. PRELIMINARIES
Definition 2.1: Let X be a non-empty set. A function $D^*: X^3 \to [0,\infty)$ is said to be a ***generalized metric*** or ***D*-metric*** on X, if it satisfies the following conditions:

(i) $D^*(x,y,z) \ge 0$ for all $x,y,z \in X$

(ii) $D^*(x,y,z) = 0$ if and only if $x = y = z$

(iii) $D^*(x,y,z) = D^*(\sigma(x,y,z))$ for all $x,y,z \in X$,
where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$

(iv) $D^*(x,y,z) \le D^*(x,y,w) + D^*(w,z,z)$ for all $x,y,z,w \in X$.

The pair (*X*, *D**), where *D** is a generalized metric on *X* is called a ***D*-metric space*** or a ***generalized metric space.***

Definition 2.2: A selfmap f of a *D**-metric space (*X*, *D**) is called a ***K-contraction***, if there is a q with $0 \le q < \frac{1}{2}$ such that

$$D^*(fx, fy, fz) \le q.\max\{D^*(x, fx, fx) + D^*(y, fy, fy)\}$$

for all $x, y \in X$

The notions of contraction and K-contraction are independent. In this thesis we define a special type of contractions called λ-generalized contractions for *D**-metric spaces as follows:

Definition 2.3: A selfmap f of a *D**-metric space (*X*, *D**) is called a λ***-generalized contraction***, if for every $x, y \in X$, there exist non-negative numbers q, r, s and t (all depending on *x* and *y*) such that

$$\sup_{x, y \in X} \{q + r + s + 2t\} = \lambda < 1 \text{ and}$$

$$D^*(fx, fy, fz) \leq q.D^*(x, y, y) + r.D^*(x, fx, fx) + s.D^*(y, fy, fy)$$
$$t.\{D^*(x, fy, fy) + D^*(y, fx, fx)\}$$

for all $x, y \in X$

III. MAIN RESULT

Theorem 3.1: suppose f is a selfmap of a *D**-metric space (*X*, *D**) and *X* be *f*-orbitally complete. If f is a λ-generalized contraction, then it has a unique fixed point $u \in X$. In fact,

(3.1.1) $u = \lim_{n \to \infty} f^n x$ for any $x \in X$

and

(3.1.2) $D^*(f^n x, u, u) \leq \dfrac{\lambda^n}{1-\lambda} D^*(x, fx, fx)$ for all $x \in X$ and $n \geq 1$.

Proof: $x \in X$ be arbitrary and define the sequence $\{x_n\}$ by $x_0 = x,\ x_1 = fx_0,\ x_2 = fx_1 = f^2 x, \ldots, x_n = fx_{n-1} = f^n x, \ldots$. Note that the orbit of *x* under *f*, $O_f(x:\infty) = \{x_n : n = 0, 1, 2, 3, \ldots\}$

Consider,

$$D^*(x_n, x_{n+1}, x_{n+1}) = D^*(fx_{n-1}, fx_n, fx_n)$$
$$\leq q(x_{n-1}, x_n) D^*(x_{n-1}, x_n, x_n) + r(x_{n-1}, x_n) D^*(x_{n-1}, fx_{n-1}, fx_{n-1})$$
$$+ s(x_{n-1}, x_n) D^*(x_n, fx_n, fx_n)$$
$$+ t(x_{n-1}, x_n)\{D^*(x_{n-1}, fx_n, fx_n) + D^*(x_n, fx_{n-1}, fx_{n-1})\}$$
$$\leq q(x_{n-1}, x_n) D^*(x_{n-1}, x_n, x_n) + r(x_{n-1}, x_n) D^*(x_{n-1}, x_n, x_n)$$
$$+ s(x_{n-1}, x_n) D^*(x_n, x_{n+1}, x_{n+1})$$
$$+ t(x_{n-1}, x_n)\{D^*(x_{n-1}, x_{n+1}, x_{n+1}) + D^*(x_n, x_n, x_n)\}$$

Writing $q_{n-1} = q(x_{n-1}, x_n),\ r_{n-1} = r(x_{n-1}, x_n),\ s_{n-1} = s(x_{n-1}, x_n)$ and $t_{n-1} = t(x_{n-1}, x_n)$, we get

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$$D^*(x_n, x_{n+1}, x_{n+1}) \le q_{n-1} D^*(x_{n-1}, x_n, x_n) + r_{n-1} D^*(x_{n-1}, x_n, x_n)$$

$$+ s_{n-1} D^*(x_n, x_{n+1}, x_{n+1}) + t_{n-1}\{ D^*(x_{n-1}, x_{n+1}, x_{n+1})$$

$$+ D^*(x_n, x_n, x_n) \}$$

$$\le (q_{n-1} + r_{n-1}) D^*(x_{n-1}, x_n, x_n) + s_{n-1} D^*(x_n, x_{n+1}, x_{n+1})$$

$$+ t_{n-1}\{ D^*(x_{n-1}, x_{n+1}, x_{n+1}) \}$$

$$\le (q_{n-1} + r_{n-1}) D^*(x_{n-1}, x_n, x_n) + s_{n-1} D^*(x_n, x_{n+1}, x_{n+1})$$

$$+ t_{n-1} D^*(x_{n-1}, x_n, x_n) + t_{n-1} D^*(x_n, x_{n+1}, x_{n+1})$$

$$\le (q_{n-1} + r_{n-1} + t_{n-1}) D^*(x_{n-1}, x_n, x_n) + (s_{n-1} + t_{n-1}) D^*(x_n, x_{n+1}, x_{n+1})$$

which implies that

$$(1 - s_{n-1} - t_{n-1}) D^*(x_n, x_{n+1}, x_{n+1}) \le (q_{n-1} + r_{n-1} + t_{n-1}) D^*(x_{n-1}, x_n, x_n)$$

And hence

$$D^*(x_n, x_{n+1}, x_{n+1}) \le \frac{(q_{n-1} + r_{n-1} + t_{n-1})}{(1 - s_{n-1} - t_{n-1})} D^*(x_{n-1}, x_n, x_n)$$

$$\le \frac{(\lambda - s_{n-1} - t_{n-1})}{(1 - s_{n-1} - t_{n-1})} D^*(x_{n-1}, x_n, x_n)$$

That is,

$$D^*(x_n, x_{n+1}, x_{n+1}) \le \lambda\ D^*(x_{n-1}, x_n, x_n), \text{ where } \lambda = \underset{x, y \in X}{Sup} \{q + r + s + 2t\}$$

Thus by iteration, we get

$$D^*(x_n, x_{n+1}, x_{n+1}) \le \lambda^n\ D^*(x_0, x_1, x_1) = \lambda^n\ D^*(x_0, fx_0, fx_0) \quad \text{--------} (A)$$

Therefore

$$D^*(x_n, x_{n+p}, x_{n+p}) \le D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, x_{n+2}, x_{n+2})$$
$$+ D^*(x_{n+2}, x_{n+3}, x_{n+3}) + \ . \ . \ . + D^*(x_{n+p-1}, x_{n+p}, x_{n+p})$$
$$\le \lambda^n\ D^*(x_0, x_1, x_1) + \lambda^{n+1}\ D^*(x_0, x_1, x_1)$$
$$+ \lambda^{n+2}\ D^*(x_0, x_1, x_1) + \ . \ . \ . + \lambda^{n+p-1}\ D^*(x_0, x_1, x_1)$$

$$\le (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ . \ . \ . + \lambda^{n+p-1} + \ . \ . \ .) D^*(x_0, x_1, x_1)$$

$$= \frac{\lambda^n}{1 - \lambda} D^*(x_0, x_1, x_1) \quad \text{------------------} (B)$$

Hence $D^*\left(x_n, x_{n+p}, x_{n+p}\right) \le \frac{\lambda^n}{1-\lambda} D^*\left(x_0, x_1, x_1\right) \to 0$ as $n \to \infty$, since $0 \le \lambda < 1$. Thus $\{x_n\}$ is a sequence in $O_f\left(x:\infty\right)$ and since X is f-orbitally complete, there exists $u \in X$ such that

$$u = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f^n x_0 = \lim_{n \to \infty} f^n x$$

To show that u is a fixed point of f, first we prove

$\lim_{n \to \infty} D^*\left(fu, fx_n, fx_n\right) = 0$. Since f is λ-generalized contraction, we have

$$D^*\left(fu, fx_n, fx_n\right) \le q\left(u, x_n\right) D^*\left(u, x_n, x_n\right) + r\left(u, x_n\right) D^*\left(u, fu, fu\right)$$

$$+ s\left(u, x_n\right) D^*\left(x_n, fx_n, fx_n\right) + t\left(u, x_n\right)\left\{D^*\left(u, fx_n, fx_n\right)\right.$$

$$\left. + D^*\left(x_n, fu, fu\right)\right\}$$

That is,

$$D^*\left(fu, fx_n, fx_n\right) \le q\left(u, x_n\right) D^*\left(u, x_n, x_n\right) + r\left(u, x_n\right) D^*\left(u, fu, fu\right)$$

$$+ s\left(u, x_n\right) D^*\left(x_n, x_{n+1}, x_{n+1}\right) + t\left(u, x_n\right)\left\{D^*\left(u, x_{n+1}, x_{n+1}\right)\right.$$

$$\left. + D^*\left(x_n, fu, fu\right)\right\}$$

$$\le q\left(u, x_n\right) D^*\left(u, x_n, x_n\right) + r\left(u, x_n\right) D^*\left(u, x_{n+1}, x_{n+1}\right)$$

$$+ r\left(u, x_n\right) D^*\left(x_{n+1}, fu, fu\right) + s\left(u, x_n\right) D^*\left(x_n, x_{n+1}, x_{n+1}\right)$$

$$+ t\left(u, x_n\right) D^*\left(u, x_{n+1}, x_{n+1}\right) + t\left(u, x_n\right) D^*\left(x_n, x_{n+1}, x_{n+1}\right)$$

$$+ t\left(u, x_n\right) D^*\left(x_{n+1}, fu, fu\right)$$

$$\le q\left(u, x_n\right) D^*\left(u, x_n, x_n\right) + \left\{r\left(u, x_n\right) + t\left(u, x_n\right)\right\} D^*\left(u, x_{n+1}, x_{n+1}\right)$$

$$+ \left\{r\left(u, x_n\right) + t\left(u, x_n\right)\right\} D^*\left(x_{n+1}, fu, fu\right)$$

$$+ \left\{s\left(u, x_n\right) + t\left(u, x_n\right)\right\} D^*\left(x_n, x_{n+1}, x_{n+1}\right)$$

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$$\le \lambda D^*(u, x_n, x_n) + \lambda D^*(u, x_{n+1}, x_{n+1})$$

$$+\lambda D^*(x_{n+1}, fu, fu) + \lambda D^*(x_n, x_{n+1}, x_{n+1})$$

That is,

$$(1-\lambda) D^*(fu, fx_n, fx_n) \le \lambda \{D^*(u, x_n, x_n) + D^*(u, x_{n+1}, x_{n+1})$$

$$+D^*(x_n, x_{n+1}, x_{n+1})\}$$

Therefore

$$D^*(fu, fx_n, fx_n) \le \frac{\lambda}{(1-\lambda)} \{D^*(u, x_n, x_n) + D^*(u, x_{n+1}, x_{n+1})$$

$$+D^*(x_n, x_{n+1}, x_{n+1})\}$$

Which implies that $\lim_{n \to \infty} D^*(fu, fx_n, fx_n) = 0$, and hence

$fu = \lim_{n \to \infty} fx_n = \lim_{n \to \infty} x_{n+1} = u$, showing u is a fixed point of *f*.

To prove that f has unique fixed point, suppose that $fx_0 = x_0$ and $fy_0 = y_0$ for some $x_0, y_0 \in X$. Then by the definition of λ-generalized contraction, it follows that

$$D^*(x_0, y_0, y_0) = D^*(fx_0, fy_0, fy_0)$$

$$\le qD^*(x_0, y_0, y_0) + rD^*(x_0, fx_0, fx_0) + sD^*(y_0, fy_0, fy_0)$$

$$+t\{D^*(x_0, fy_0, fy_0) + qD^*(y_0, fx_0, fx_0)\}$$

$$= qD^*(x_0, y_0, y_0) + rD^*(x_0, x_0, x_0) + sD^*(y_0, y_0, y_0)$$

$$+t\{D^*(x_0, y_0, y_0) + qD^*(y_0, x_0, x_0)\}$$

$$= (q+2t) D^*(x_0, y_0, y_0)$$

$$\le \lambda D^*(x_0, y_0, y_0)$$

This implies that $D^*(x_0, y_0, y_0) = 0$, since $\lambda < 1$, and hence $x_0 = y_0$. Thus f has unique fixed point.Since x is arbitrary in the above discussion, it follows that $u = \lim_{n \to \infty} f^n x$ for any $x \in X$ and hence

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(2.2.2) is proved. Finally, since $D^*(x_n, x_{n+p}, x_{n+p}) = \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx)$ (by (B)), on letting $p \to \infty$, we get

$$D^*(x_n, u, u) = \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx),$$ proving (2.2.3). This completes the proof of theorem.

3.2.**Corollary:** Suppose f is a selfmap of a *D**-metric space (*X*, *D**) and *X* is f-orbitally complete. If f is a contraction of (*X*, *D**), then it has a unique fixed point $u \in X$. In fact,

(3.2.1) $u = \lim_{n \to \infty} f^n x$ for any $x \in X$

and

(3.2.2) $D^*(f^n x, u, u) \le \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx)$ for all $x \in X$ and $n \ge 1$.

Proof: In view of the fact that, every contraction is λ-generalized contraction, Corollary follows from Theorem 3.1.

3.3.**Remark:** The Banach contraction principle is a particular case of Corollary 3.2.In fact, if (X, *d*) is a complete metric space, then by Corollary 1.10.2, (X, D_1^*) is a complete *D**-metric space and hence *f*-orbitally complete for any selfmap f of *X*. Also if f is a contraction of (*X*, *d*), then the condition of contraction can written as

$$D_1^*(fx, fy, fy) \le q.D_1^*(x, y, y) \text{ for all } x, y \in X,$$

since $D_1^*(x, y, y) = d(x, y)$; so that f is a contraction on (X, D_1^*). Thus f is a contraction on the f-orbitally complete *D**-metric space (X, D_1^*) and hence the conclusions of Corollary 2.2.4 hold for f; which are the conclusions of the Banach contraction principle.

3.4.**Corollary:** Suppose f is a selfmap of a *D**-metric space (*X*, *D**) and *X* is f-orbitally complete. If f is a K-contraction of (*X*, *D**) with constant q, then it has a unique fixed point $u \in X$. In fact,

(3.4.1) $u = \lim_{n \to \infty} f^n x$ for any $x \in X$

and

(3.4.2) $D^*(f^n x, u, u) \le \frac{2q^n}{1-2q} D^*(x, fx, fx)$ for all $x \in X$ and $n \ge 1$.

Proof: In view of the fact that, every K-contraction is λ-generalized contraction, Corollary follows from Theorem 2.2.1 by taking $\lambda = 2q$.

Remark: Kannan's result is particular case of the Corollary 3.2.In fact, if (X, *d*) is a complete metric space, then by Corollary 3.3, (X, D_1^*) is a complete *D**-metric space and hence *f*-orbitally complete for any selfmap f of *X*. Also if f is a K-contraction, with constant q, of (*X*, *d*), then the condition of K-contraction can be written as

$$D_1^*(fx, fy, fy) \le q\{D_1^*(x, fx, fx) + D_1^*(y, fy, fy)\} \text{ for all } x, y \in X,$$

since $D_1^*(x, y, y) = d(x, y)$; so that f is a K-contraction on (X, D_1^*). Thus f is a K-contraction on the f-orbitally complete *D**-metric space (X, D_1^*) and hence the conclusions of Corollary 3.2 hold for f; which are the conclusions of the Kannan's result.

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